ANALYSIS OF NOTCHES AND CRACKS: PROGRESS IN PILOT COMPARISONS BETWEEN EXPERIMENT AND THEORY

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ABSTRACT

Three preliminary studies in progress to assess the correlation to be expected between elastic-work-hardening-plastic continuum theory and experiments are reviewed. Consideration is being given to required accuracy of the specification of the stress-strain curve and appropriateness of assumption of small strains and rotations in the theory. A previously developed finite element computer program is being used, and experimental data has been obtained from other laboratories. The studies include plastic deformation about an initially round hole cross-bored through a 2024-T35 aluminum alloy rod subjected to pure twist, local behavior in plane strain near a crack tip in a thick plate, and thickness reduction in plane stress near a crack tip in a thin sheet. These studies are intended to provide the experience required to evaluate adequately a sophisticated finite difference computer program for elastic-work-hardening-plastic analysis of solids containing notches or cracks. The development of such a program is in progress under this grant.

HGMcComb:eks (Typed 2-29-68)

FOREWORD

This report describes work performed in the Department of Mechanical Engineering at the Carnegie Institute of Technology of Carnegie-Mellon University for Langley Research Center, National Aeronautics and Space Administration, under NASA grant NGR-39-002-023, "Analysis of Notches and Cracks." The work was performed between October 1966 and December 1967. Notes for this report are kept in File SM-8.

It is a pleasure to acknowledge the cooperation of Mr. H. G. McComb, Technical Liaison Officer at Langley Research Center, during the tenure of this research. Valuable technical contributions have also been made by Mrs. Carol Ann Clark, Dr. T. A. Cruse, and Messrs. D. Douge and R. R. Shuck. Thanks are due to Miss Judith Kajder for her meticulous preparation of the manuscript.

I. INTRODUCTION

The work being done in the current research program involves generating solutions to the equations of elasto-plastic flow. Such solutions may be distinguished from earlier work of a similar nature by two factors. The first is that the numerical techniques under development[1] must be of inherently high accuracy. Thus we are designing methods which should produce errors of the order of one per cent. The second factor is that the solutions so derived must be physically meaningful, at least for problems of a certain type.

Interwoven with these characteristics is the objective of applying the solution techniques to problems involving notches and cracks in otherwise simple shapes, e.g., rods and plates. Viewed as initial- and boundary-value problems in elasto-plasticity theory, the analyst finds no obstacle in principle to obtaining a myriad of solutions. (Operationally, of course, there are many difficulties, the resolution of which has occupied considerable effort over the last year.) Less clear, however, is the requirement of being physically meaningful. To begin with, it has not been established fully whether the theory of elasto-plastic flow is an accurate model of the associated physical event. Should this be the case, we do not know precisely what conditions need be met to use this model. Two items are of particular interest. The first concerns how accurately the local stress-strain curve may be specified, and the second concerns whether small strain and rotation assumptions are appropriate.

Commenting briefly on the stress-strain curve, we may note the availability of three forms for a given material test. There is first the "engineering" stress-strain curve which is merely a rescaled force-displacement curve taken from a simple laboratory test, say, under uniaxial tension. There is the so-called true stress-true strain curve which accounts for incompressibility of plastic flow. Finally, (Bridgman's) correction may be applied to account for localized necking near the end of the test, prior to fracture of the specimen. So far as we know, these forms may provide a basis for selecting a reasonable representation of the actual - as opposed to "true" - stress-strain characteristic of the material. To the writer's knowledge, however, the (quantitative) accuracy of each of these curves has not been established. This situation becomes problematic in the event nearly exact solutions to the original field equations are available; errors could accrue from incorrect material characteristics.

Closely allied to this ambiguity is the matter of how a uniaxial curve, even if properly specified, may be generalized to a triaxial stress state. This is the task of selecting an equivalent stress, e.g., octahedral or maximum shear. Without going into further detail, it is pertinent to note that proper characterization of a given (metallic) material thus remains an open, important, and not fully studied problem.

The tack taken in the present program is, in a sense, to evaluate the state of the art via suitable pilot studies. We thus use solution methods developed previously[2-4] in conjunction with experimental work performed at other laboratories by specialists in this phase of the work. The viewpoint therefore is, given the indeterminacy of the model and the material characterization, can we make any quantitative assessment of the degree of compatibility between theory and experiment? Of course, such a view implies that we learn how to make the comparisons, and that we begin the arduous task of establishing elasto-plasticity as a predictive device.*

Three separate problems have been selected for these pilot studies; progress in each is discussed in the following sections. It should be noted that, for all, considerable effort is being made to distinguish between numerical errors inherent in the solution method, and modelling errors that accrue as noted above. Hence, a fair amount of attention is directed towards the manner in which the numerical solution is to be interpreted.

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*It is interesting to compare the state of the art of plasticity, i.e., elasto-plastic flow theory, in this respect with that of elasticity. The former is in its infancy, the latter fully established.

II. PLASTIC DEFORMATION OF A HOLE IN SHEAR

The problem to be studied arises from an experimental study performed by Alpaugh.[5] He used a 1.75 in dia rod of 2024-T35 aluminum that was cross-bored with a hole 0.0625 in dia. The rod was loaded in a torsion machine and loading continued until the applied shear strain was about 0.20.

The companion analysis was performed in the following manner. We presume a square specimen, whose half-width is 1 in, under condition of plane strain. The specimen contains a circular hole of radius 0.1 in through its center. The specimen is loaded in compression on its upper and lower faces and in tension of the same amount on its remaining faces. In this manner, the loading is equivalent to the pure shear of a torsion machine.

Loading proceeded in 39 increments as follows. The first increment was adjusted so that yielding was ready to begin at the most highly stressed portions of the body (16,000 lb/in^2). Subsequent load increments were made very much smaller, and the total load achieved was 50,000 lb/in^2.

In as much as the problem has certain symmetry, we need only consider one quadrant of the square. This was divided into 200 triangular elements which, although a fairly coarse mesh, was thought to be adequate to the purposes at hand. Vertices of the elements met to define 121 nodes which, for constant-strain finite elements, gave 242 degrees of freedom.

At the end of each load increment, the displacements of each nodal point were (algebraically) added to the coordinates of that node.

In this manner, the changing shape of the body was taken into account, albeit approximately.

The original and final shapes thus obtained are shown in Figure 2.1; note that the x-axis on this Figure corresponds to a 45° helix on the torsion specimen.

The eccentricity of the hole may be defined as

$$m = (A - B)/(A + B) \tag{2.1}$$

where A and B are the major and minor axes of the hole. From the theory of elasticity, it may be shown that

$$m = 4(1 - \nu)\varepsilon_x \tag{2.2}$$

where ν is Poisson's ratio ($\nu = 1/3$ in this case) and ε_x is the strain (in the x direction in Figure 2.1) far from the hole. McClintock et al.[6] predicted from simplified analysis that

$$m = 2\varepsilon_x \tag{2.3}$$

whereas a straight-line fit of Alpaugh's data indicated that

$$m = 3.26\varepsilon_x \tag{2.4}$$

The question therefore becomes, what relation between m and ε_x is given by the finite element analysis? The answer is to be seen in

Figure 2.2. The early stages of loading agree with the elastic solution, given in (2.2). Very quickly, however, the behavior changes to conform more nearly to Alpaugh's data and continues up to a load of 45,000 lb/in^2, after which the two diverge.

Comment: To a large extent, comparison between theory and experiment is most encouraging. We have, after all, predicted with reasonable accuracy the relation between eccentricity m and "applied" strain ε_x. Note that ε_x is one-half the shear strain applied by Alpaugh. Still, there are certain deficiencies in the solution which, for the most part, appear attributable to the use of the finite element method as a basic solution technique. These same characteristics appear in other problems discussed in this report and will not be shown in detail here. Usually, these characteristics involve non-uniformities in the stress gradients in a few areas of high stress. For example, if the stress components σ_x, σ_y, and τ_{xy} are plotted as a function of angular position aroung the hole periphery,* they do not vary smoothly with position. For the first (elastic) load increment, the stresses show errors - compared to the theoretical results - of roughly constant error from one element to the next, but of alternating sign. This trend tends to be accentuated as yielding proceeds.

A second shortcoming is the deviation between theory and experiment at higher loads - see Figure 2.2. There are several factors, of course, that could mitigate the deviation. Alpaugh's data are noticeably sparse above $\varepsilon_x = 0.08$ (two points above this strain level, thirteen

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* Specifically we should like to plot $\sigma_x(r,\theta) = (a,\theta)$, etc. In lieu of having data at precisely $r = a$, we would graph stresses for elements bordering the hole.

below), so that it is not a certainty that (2.4) represents his results at high strains. On the other hand, the growth in ε_x per unit change in (stress) loading is greatest at this level, so that the numerical data become suspect. Two possibilities are evident. The first is that the load increment was too large in this phase. The second is that the means used to account for the changing shape of the hole becomes too approximate in this strain range. This effect may occur either because the basic approximation is invalid in this range or because the use of a "true" stress-strain curve is inappropriate.

Were the thrust of the overall program to investigate this particular problem, considerably further work would be indicated. It is believed, however, that the primary objective has been attained, namely, to ascertain the utility of elasto-plastic theory as a predictive device. The numerical and experimental data are remarkably consistent. It further appears that improvements can be effected by suitable manipulation of the finite element technique but that such efforts are not justified at this point.[2]

III. LOCAL BEHAVIOR IN PLANE STRAIN

As a second problem area, we are examining the growth of plastic zones in "plane strain" specimens of high strength materials, notably steel. We consider a planar disc of radius r_o, having a crack that emanates from the center to the boundary. Loadings are limited to those symmetric above and below the crack line, so that only half the disc need be studied - see Figure 3.1. The disc is divided into 525 elements and has 289 nodes, or 578 degrees of freedom. The disc is regarded as having been cut out from, say, a tensile specimen so that it represents the immediate vicinity of the crack tip. This view is further enhanced by two additional measures. First we set $r_o/b = 0.1$, where b is the half-crack length. Secondly we impose loading on $r = r_o$ derived from the one-term singularity solution:

$$\sigma_r = \bar{\sigma}\sqrt{b/2r}\ (\frac{5}{4}\cos\psi/2 - \cos 3\psi/2)$$

$$\sigma_\psi = \bar{\sigma}\sqrt{b/2r}\ (\frac{3}{4}\cos\psi/2 + \cos 3\psi/2) \qquad (3.1)$$

$$\tau_{r\psi} = \bar{\sigma}\sqrt{b/2r}\ (\frac{1}{2}\sin\psi/2 + \frac{1}{2}\sin 3\psi/2)$$

where $\bar{\sigma}$ is the uniform stress imposed on the plate an infinite distance away.

To date, we have checked the elastic solution only and have found the numerical results to be in reasonable agreement with theory. In Figures 3.2 - 3.4 are plotted $\sigma_y/\bar{\sigma}$, $\sigma_x/\bar{\sigma}$, and $3\tau_o/2\bar{\sigma}$ at $r = 0.0008b$, as functions of angular position. The pattern of alternating high-low results is manifest but is not expected to be particularly harmful. Of special interest is the relative accuracy of the stress concentration, particularly in comparison to previous results.[4]

Using this array of elements we plan to proceed to higher loads, i.e., such that yielding occurs and fills a reasonable portion of the disc. The stress-strain curve will be that of a high-strength alloy which experimentally has been observed to follow the predictions of linear fracture mechanics. Such behavior would appear to justify the formulation of this problem.

Comment: One of the interesting features of the analytical solution is that the radial displacement vanishes at $\psi = \pi$, viz.

$$u_r = (\bar{\sigma}/4\mu)\sqrt{(br/2)}\,[(5-8\nu)\cos\psi/2 - \cos 3\psi/2]$$

$$u_\psi = (\bar{\sigma}/4\mu)\sqrt{(br/2)}\,[(5-8\nu)\sin\psi/2 - \sin 3\psi/2] \qquad (3.2)$$

where μ is the shear modulus and ν is Poisson's ratio. This characteristic is not observed in the finite element solution in which, for $\psi \rightarrow \pi$, u_r is small but not zero. As a consequence the finite element solution does not reproduce the high stress gradients in the circumferential direction as $\psi \to \pi$ (see Figures 3.2 - 3.4). This behavior is not of critical importance, as yielding is not expected to be large in this area.

This study can have significance beyond its present purpose. One might ask, for example, the load level at which yielding causes displacements to deviate markedly from the values predicted by (3.2). If this loading is greater than the known fracture stress, some light may be shed on the reasons K_{IC} is a useful parameter for this class of materials. Many other possibilities come to mind as well; it is anticipated that the utility of this problem study will be far greater than the comparison between theory and experiment originally contemplated.[9]

IV. STRAIN MEASUREMENTS IN PLANE STRESS

A third problem area is the outgrowth of a presentation at a recent ASTM Subcommittee meeting. Underwood and Kendall[10] have begun measurements in single edge-notched sheets of copper, pin-loaded in tension. They use optical interferometry to measure thickness reduction strain (ε_z) and the moiré method to determine strain (ε_y) in the direction of load application. The result, of course, is an accurate determination of these two strain components in a simple geometric shape. The question raised thereby is whether their experimental results are predictable by finite-element elaso-plastic analysis.

To date, two attempts have been made to provide an affirmative answer to this question. Without rehearsing the various details of the progress of this work, we note the conclusion drawn so far.

Of the three major components of problem formulation - geometry, loading, and material characteristics - only the first is easily specified. The loading, particularly that due to a pin, is not a trifling matter. The load distribution on a line parallel to, but several crack lengths away from, the crack is distinctly non-uniform. It turns out, however, that imposition of uniform tensile loading does not alter the strain field in a major fashion, at least in this problem. Such a conclusion, of course, is merely a restatement either of St. Venant's principle or the basic fact of linear fracture mechanics, that the presence of the crack overwhelms all lesser features.

On the other hand, the results are noticeably sensitive to proper specification of the material characteristics, i.e., the elastic constants and the stress-strain curve, as yielding progresses. In the first attempt to match experiment and numerical results, the discrepancy was approximately an order of magnitude. This was traced to a significant over estimation of the proportional limit. Correcting this required performance of stress-strain tests of considerable care, particularly in the region of initial yield. The second attempt was made on this basis. The results are suggested by Figures 4.1 and 4.2 which show strains along the line of crack prolongation. While the general behavior is similar, the numerically predicted strains are noticeably below the experimental data. Certainly some improvement over the first attempt is obvious, but the comparison remains far from satisfactory.

Upon detailed examination of these results, two factors emerge as being worth further investigation. The first is a continuation of the specification of the stress-strain curve in the region of initial yielding. The stress-strain curve of copper is known to exhibit but little linear elastic strain, and extraordinary care is required to define its shape in the region of initial yielding. Further tests have just been completed; they show a dimunition of the proportional limit from 3,700 lb/in^2 to about 2,900 lb/in^2. Incorporating this correction into the analysis should produce considerable improvement in comparisons such as those shown in Figure 4.1 and 4.2.

The second factor is essentially an experimental error in determining ε_z. The method used heretofore essentially subtracts out any uniform component, i.e., that part of ε_z which is everywhere the same over the plate. As a result the experimental curve of Figure 4.2 should be shifted upwards by a small but unknown amount. Steps are in process of being taken to correct this error, which should provide improvements in the comparison.

One final item should be noted. To insure that the discrepancies in Figure 4.1 and 4.2 do not accrue from numerical approximation in the finite element technique, we made a comparison between elastic analysis and photoelastic study of comparable situations. Typical of the results are the plots shown in Figures 4.3 and 4.4 which have both theoretical and experimental data. It is to be observed that agreement is satisfactory, and that the occasional alternating errors noted above are present.

Comment: Of the three problems discussed so far, this is perhaps the most complex and has provided the most difficulty and the most useful information, viewed as a pilot study. It has shown that variations in the load distribution do not exert a major influence on the strain field near the crack tip; that experimental and theoretical interpretations of quantities selected for comparison must be amenable to comparison; and that the results can be very sensitive to the shape of the stress-strain curve, especially in the region of initial yield.

The last point is discussed more fully in the next section. The second is superficially obvious, but is rarely recognized in actual practice. Altogether too often the experimentalist and analyst use the same terms to describe quantities that are either basically different, or are similar but not precisely the same. Extrapolating on this point, it is becoming increasingly clear that the detailed comparisons anticipated in later stages of this program must be designed in such a manner that the problem specifications for both experiment and analysis tally precisely. Where differences between the two must occur, as for example in the presumed analogue between thick plates and a state of plane strain, the consequences must be independently assessed. Where possible, the experiment should be designed to minimize the influence of such differences.[11]

V. EFFECT OF INITIAL YIELDING: MODEL STUDY

An interesting and pertinent question that arises in the analysis of elasto-plastic flow concerns the shape of the stress-strain curve in the area of initial yielding. If, for example, it should happen that these local details influence the strain (or stress) field some distance from the region where they are specifically appropriate, or if they affect the subsequent field, i.e., at higher loading, then two observations are in order. The first is the particular attention should be given to prescribing this part of the stress-strain curve precisely when physical predictions are to be made. The second is that the theoretical model may have an inherent limitation, namely, substituting linear and non-linear stress-strain relations for recoverable and unrecoverable strains.

While these two points have been suggested previously,[12,13,3] an appropriate model study has not, to the writer's knowledge, been completed. It is possible, however, to investigate the matter, and an analysis is now in progress.

Consider an infinite elasto-plastic medium containing a spherical void, of radius a. If the surface of the void is subject to a pressure $p = p(t)$, where t is time, then the medium possesses spherical symmetry and the loading conditions become

$$\begin{aligned} &\sigma_r(a,t) = -\,p(t) \\ &\sigma_r(r,t) \to 0 \qquad \text{as } r \to \infty \end{aligned} \tag{5.1}$$

Spherical symmetry requires that the angular coordinates ψ, θ be null in the usual sense, and that

$$\sigma_\psi = \sigma_\theta ; \quad \sigma_{r\psi} = \sigma_{\psi\theta} = \sigma_{\theta r} = 0$$

$$\varepsilon_\psi = \varepsilon_r ; \quad \varepsilon_{r\psi} = \varepsilon_{\psi\theta} = \varepsilon_{\theta r} = 0 \tag{5.2}$$

The only active displacement component is u, in the radial direction.

The equations of equilibrium are

$$\partial\sigma_r/\partial r + 2(\sigma_r - \sigma_\psi)/r = 0 \tag{5.3}$$

and the strain-displacement relations reduce to

$$\varepsilon_r = \partial u/\partial r$$

$$\varepsilon_\psi = \varepsilon_\theta = u/r \tag{5.4}$$

For an equivalent stress equal to the octahedral stress, the equations of elasto-plastic flow[14] give the following constitutive relations

$$\overset{\circ}{\sigma}_r/2\mu = \frac{1-\nu}{1-2\nu}\,\overset{\circ}{\varepsilon}_r + \frac{2\nu}{1-2\nu}\,\overset{\circ}{\varepsilon}_\psi + \frac{2}{3}\,\frac{M}{1+M}\,(\dot{\varepsilon}_\psi - \dot{\varepsilon}_r)$$

$$\overset{\circ}{\sigma}_\psi/2\mu = \frac{\nu}{1-2\nu}\,\dot{\varepsilon}_r + \frac{1}{1-2\nu}\,\overset{\circ}{\varepsilon}_\psi - \frac{1}{3}\,\frac{M}{1+M}\,(\dot{\varepsilon}_\psi - \dot{\varepsilon}_r) \tag{5.5}$$

where μ is the elastic shear modulus, the dot denotes a time derivative, and

$$M = \mu/3\mu_{eq}^{(p)}$$

the notation following from Reference 14.

Taking the time derivative of (5.3) and (5.4), putting the latter into (5.5), and that result in to (5.3), we find

$$\frac{\partial}{\partial r}\left\{\left[\frac{1-\nu}{1-2\nu} - \frac{2}{3}\frac{M}{1+M}\right]\left[\frac{\partial \dot{u}}{\partial r} + 2\frac{\dot{u}}{r}\right]\right\} = 0 \tag{5.6}$$

as the Navier equation for this problem. The first term in brackets may be removed if we consider the stress-strain curve to be trilinear, i.e., composed of three straight-line segments as in Figure 5.1. Each segment is of constant slope in the range of r to which it is appropriate and thereby may be removed from (5.6).

The solution of (5.6) may then be shown to take the general form

$$\dot{u} = \frac{1-2\nu}{E}\dot{A}r + \frac{1+\nu}{E}\dot{B}\,r^{-2} \tag{5.7}$$

so that

$$\dot{\varepsilon}_r = \frac{1-2\nu}{E}\dot{A} - 2\,\frac{1+\nu}{E}\dot{B}\,r^{-3}$$

$$\dot{\varepsilon}_\psi = \frac{1-2\nu}{E}\dot{A} + \frac{1+\nu}{E}\dot{B}\,r^{-3} \qquad (5.8)$$

$$\dot{\sigma}_r = \dot{A} - \frac{2}{1+M}\dot{B}\,r^{-3}$$

$$\dot{\sigma}_\psi = \dot{A} + \frac{1}{1+M}\dot{B}\,r^{-3}$$

It is to be noted that (5.7) and (5.8) may be integrated with respect to time merely by removing the dots in these equations, for null initial conditions.

The stressed medium is then regarded as comprising three sections, each appropriate to its own section of the stress-strain curve. The full solution is then pieced together from three parts by observing suitable continuity conditions. The first of these is that the displacement u shall be continuous. The second is that the stress-strain curve is continuous or, equivalently, that $\tau_o = \sqrt{2}(\sigma_\psi - \sigma_r)/3$ is continuous.

Inserting these conditions, as well as (5.1), gives the result desired. These formulae are currently under process of numerical evaluation; the preliminary indication is that the character of the initial yielding will have some influence on the stress and strain fields in the manner expected. The character and degree of influence awaits quantitative evaluation which should provide useful information for making comparisons between theory and experiment.

VI. CONCLUDING REMARKS

The three studies described, together with supporting analyses, are providing useful information concerning the manner in which comparisons between theory and experiment may be made in the analysis of elasto-plastic flow near notches and cracks. Although much of the intended work awaits completion, information already derived is most instructive as to the character of result that ultimately may be anticipated.

In addition to the specific problems described, some consideration is being given to others, notably one based on the slow-bend notch geometry as used in ASTM studies.[15] This project has not been carried far enough to warrant much comment at this writing, except to note that the finite element model appears ill-suited to problems in which in-plane bending plays a major role, and thereby further work may not be performed.

In all, however, progress is deemed both satisfactory and rewarding. It is further expected that, as each of these projects is completed, a detailed individual report will be issued.

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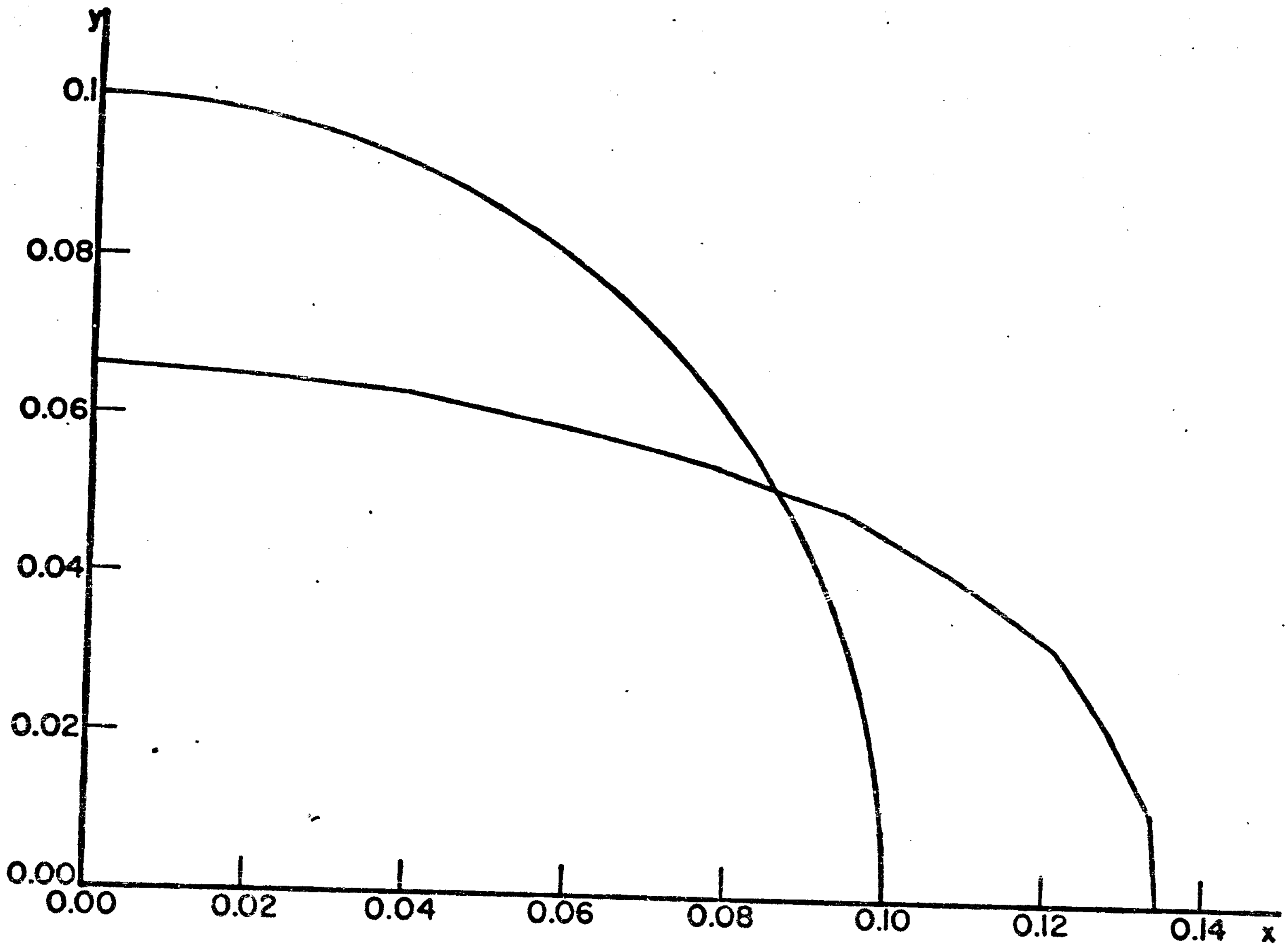

Figure 2.1a - Original and final shapes of hole in elasto-plastic body under shear (only first quadrant shown).

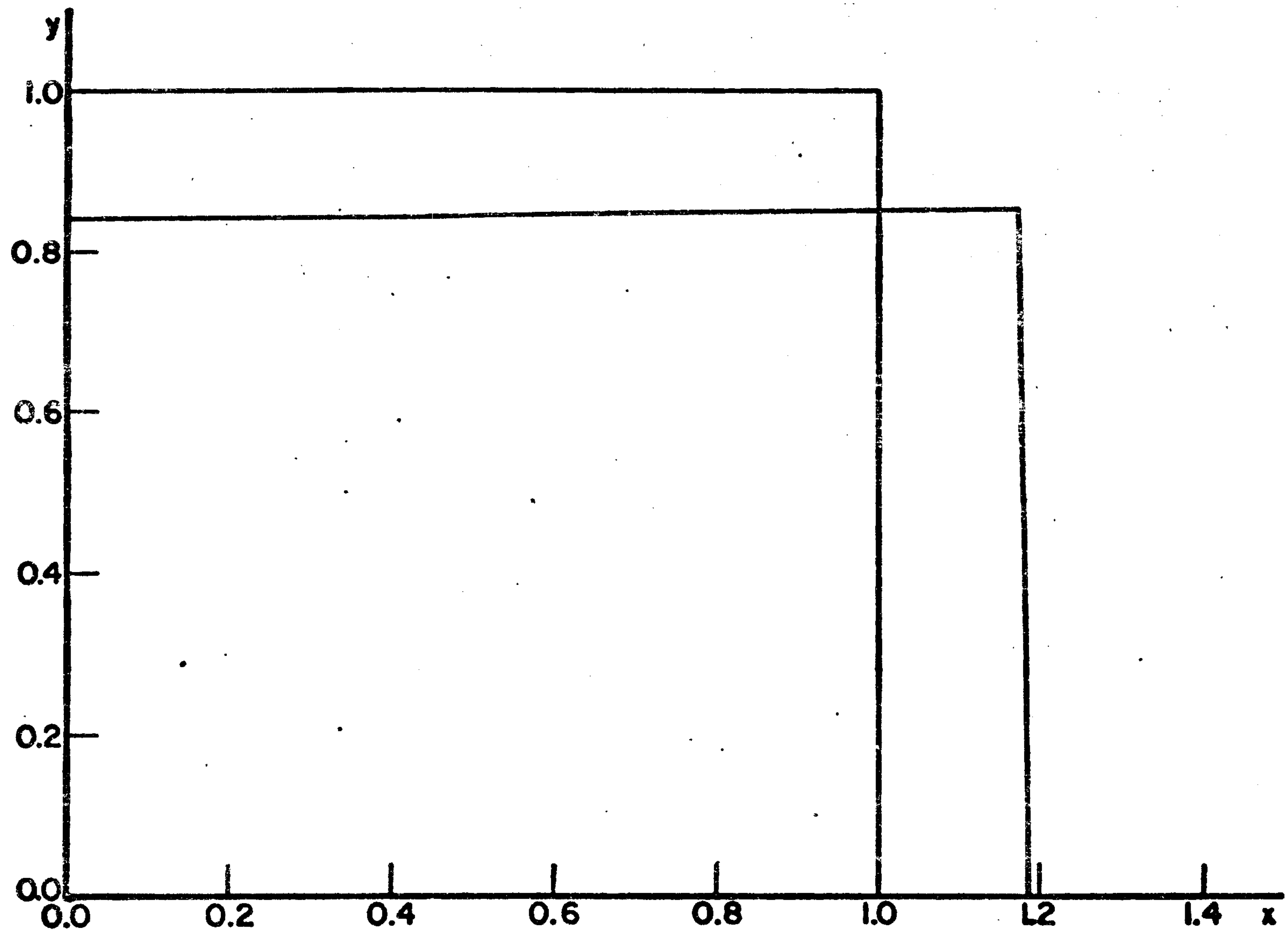

Figure 2.1b - Original and final shapes of square shape in elasto-plastic body under shear (only first quadrant shown)

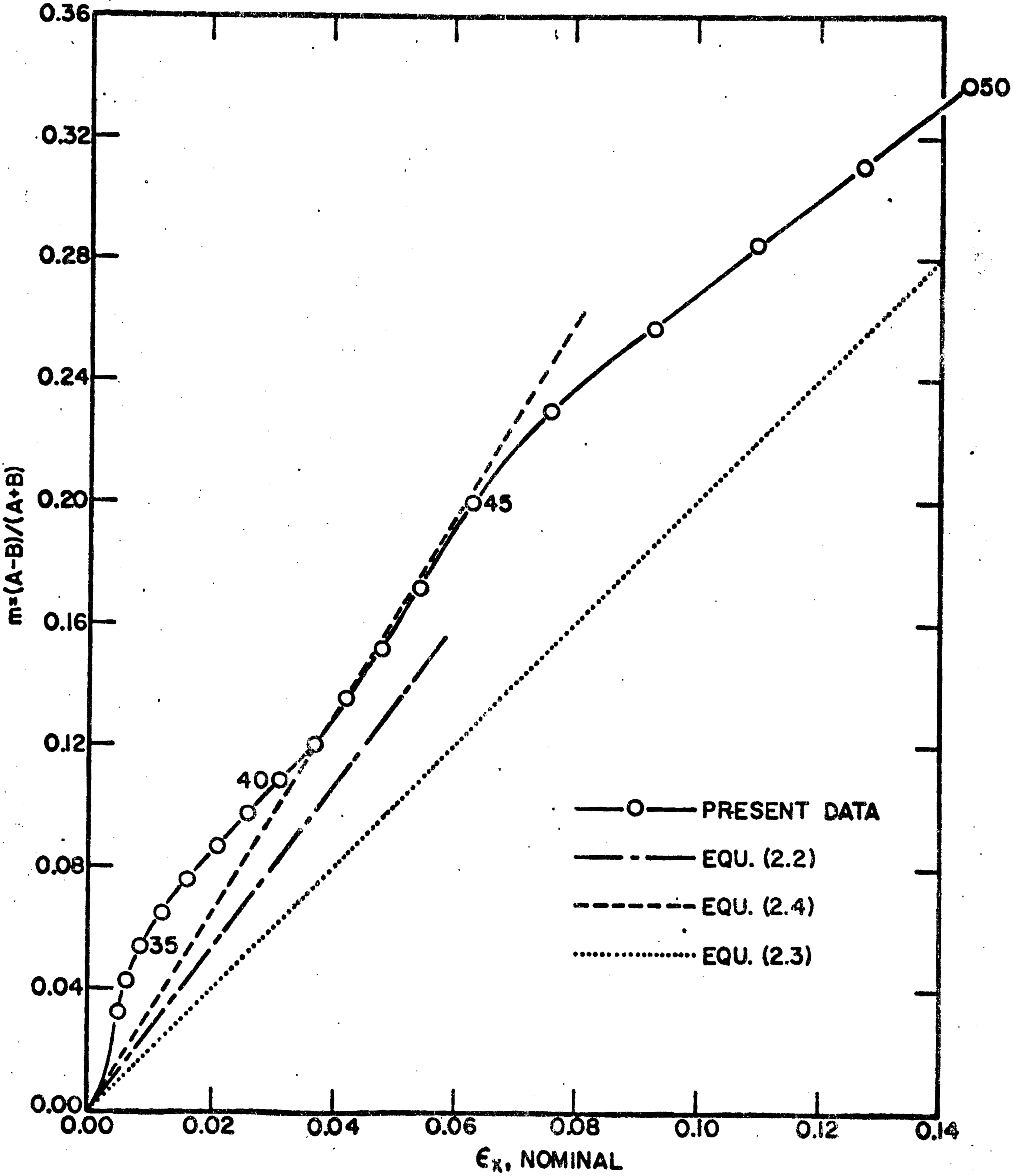

Figure 2.2 - Variation in eccentricity of hole with "applied" strain.

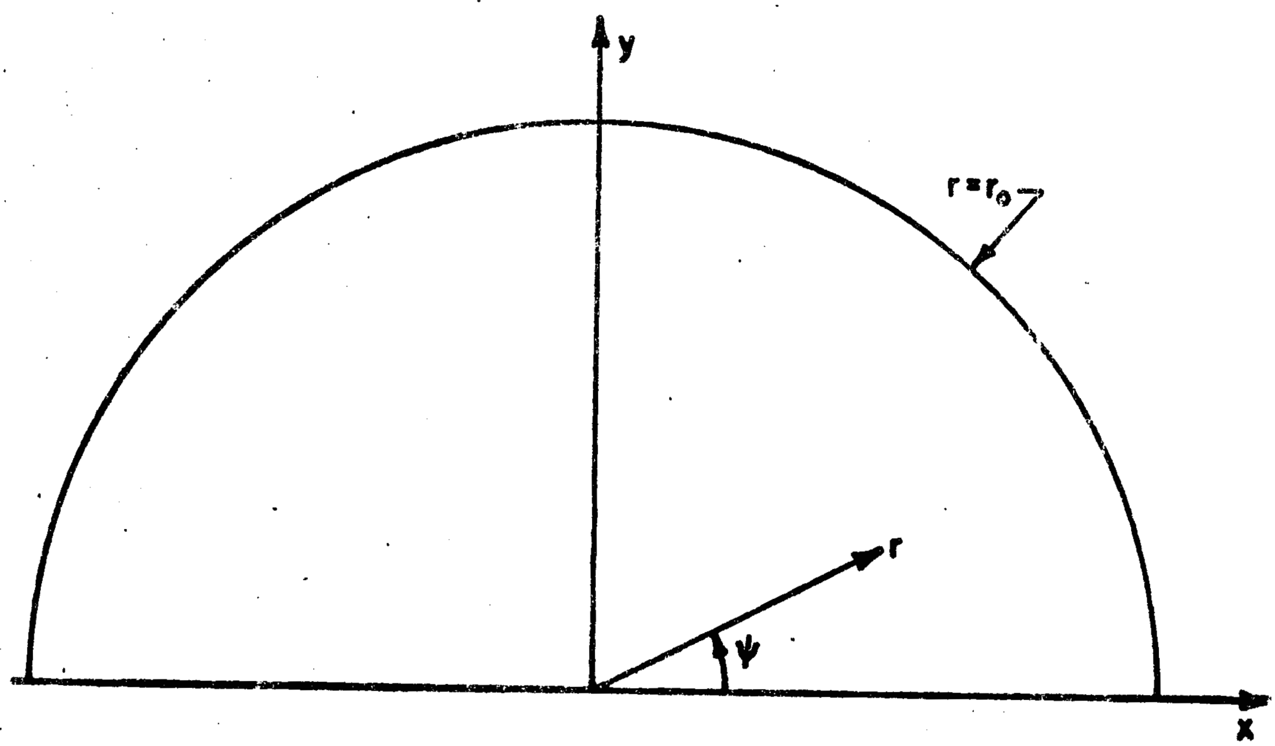

Figure 3.1 - Planar disc and coordinates - crack lies along $\psi = \pi$.

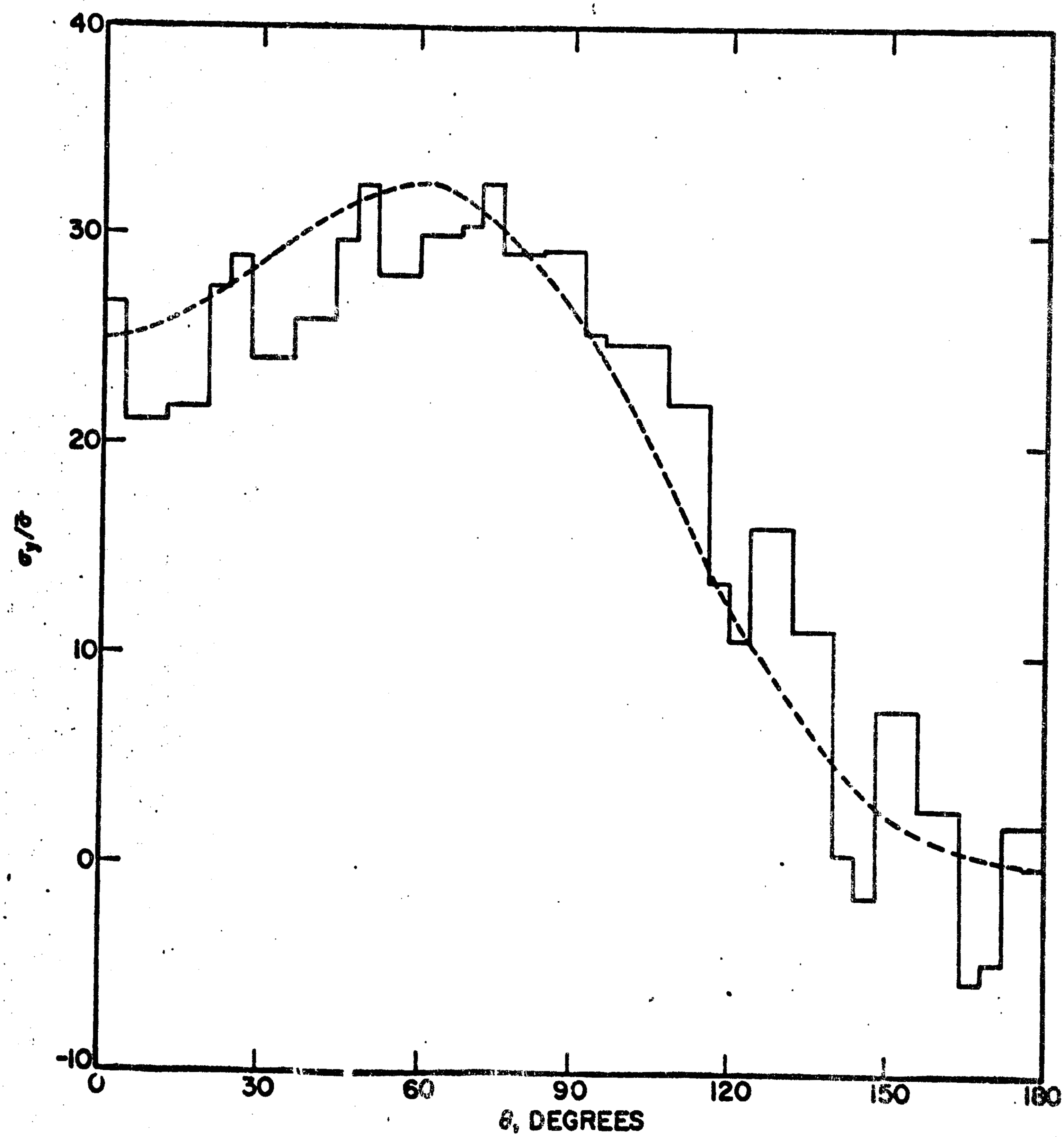

Figure 3.2 - Comparison of theoretical and numerical values of $\sigma_y(0.0008\ell_0,\theta)/\bar{\sigma}$.

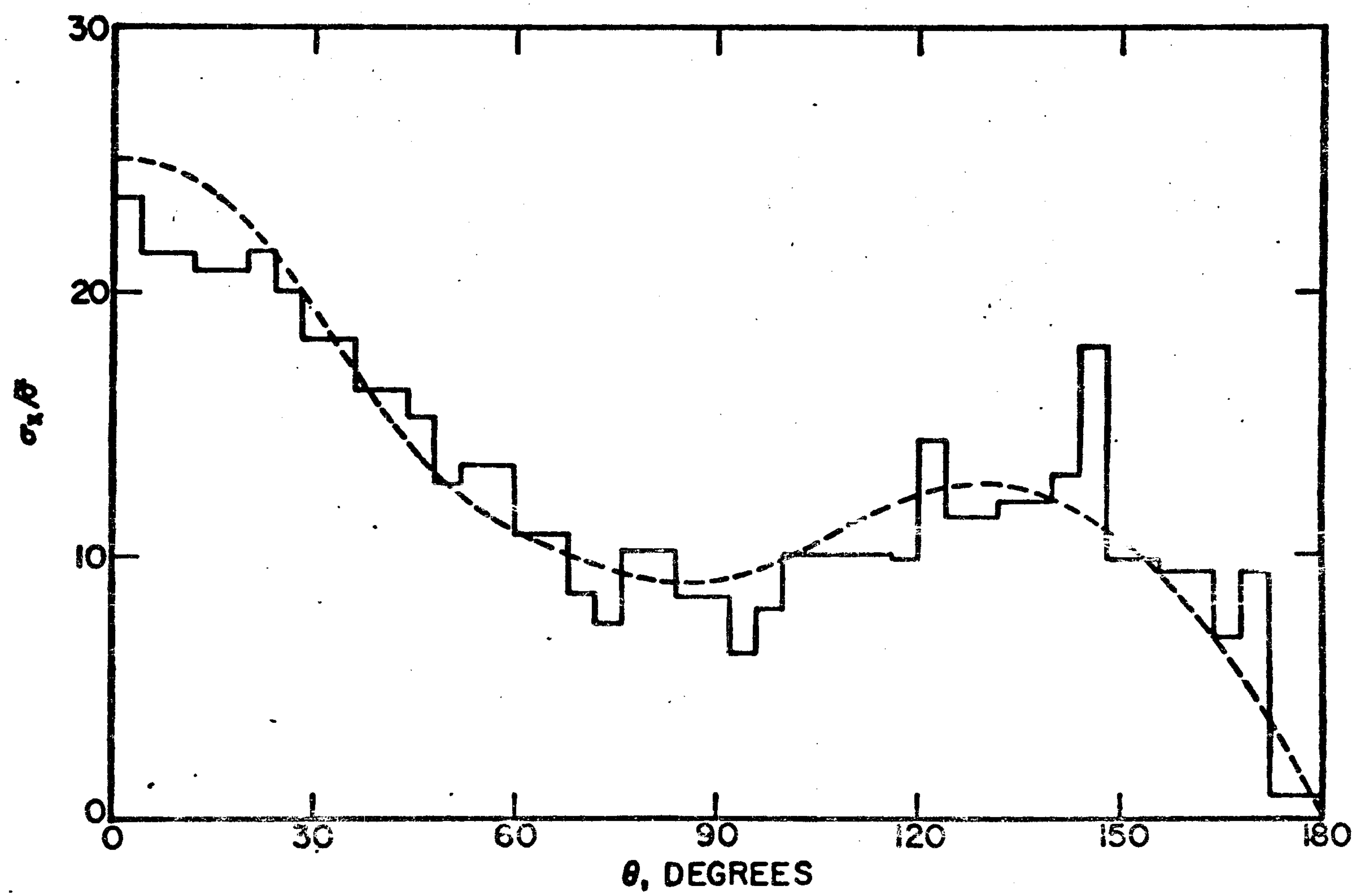

Figure 3.3 - Comparison of theoretical and numerical values of $\sigma_x(0.0008b,\theta)/\bar{\sigma}$.

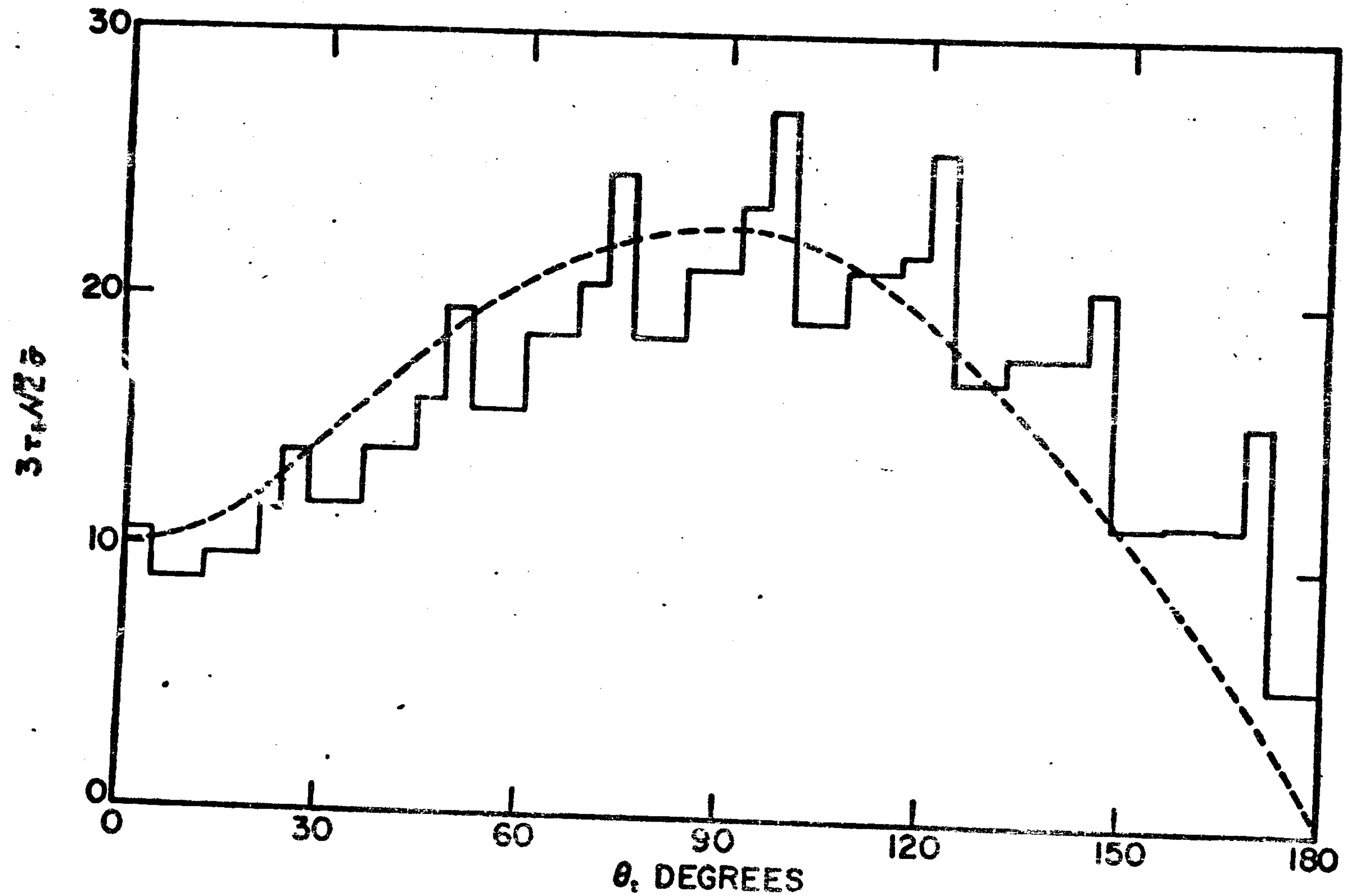

Figure 3.4 - Comparison of theoretical and numerical values of $3\tau_{\alpha}(0.0008b, \theta)/\sqrt{2\sigma}$

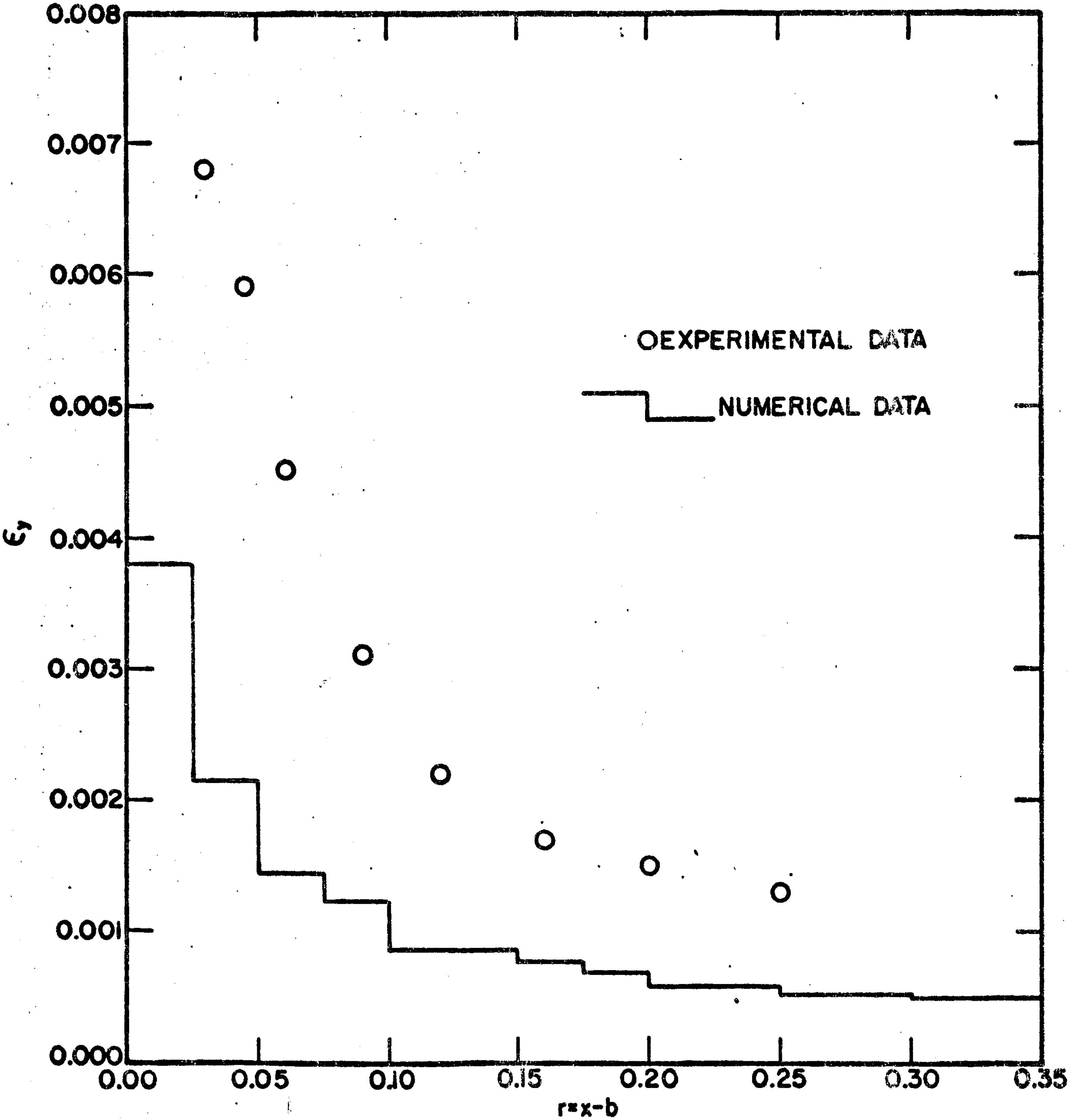

Figure 4.1 - Comparison of numerical and experimental values of $\epsilon_y(r,0)$ at $\bar{\sigma}$ = 5,200 lb/in^2 (= 1.4 x prop. lim.) in SEN copper specimen.

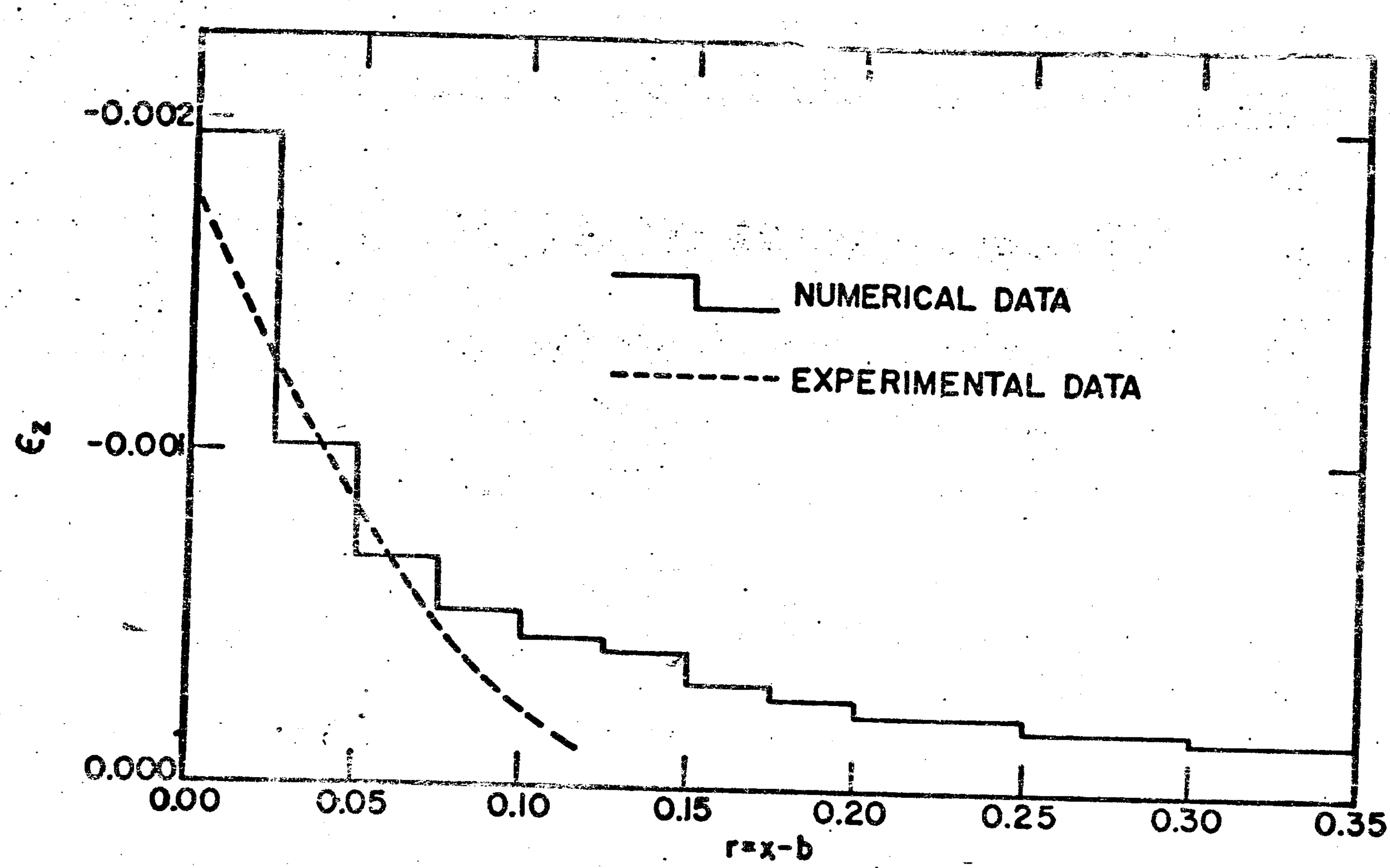

Figure 4.2 - Comparison of numerical and experimental values of $\epsilon_z(r,0)$ at $\overline{\sigma} = 5,200$ lb/in^2 (= 1.4 x prop. lim.) in SEN copper specimen.

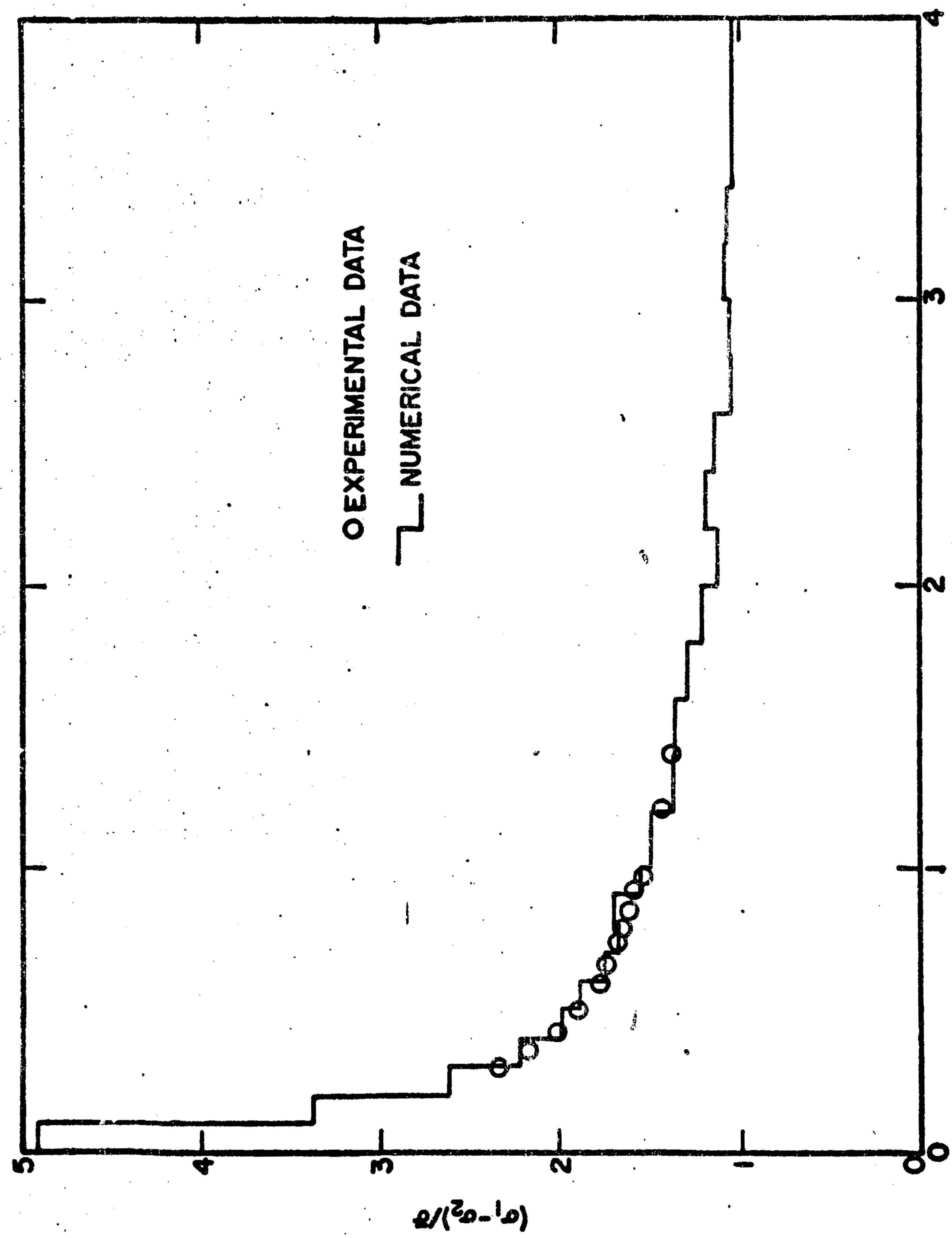

Figure 4.3 - Comparison of (elastic) numerical and photoelastic values of $2\tau_{max}/\bar{\sigma} = (\sigma_1 - \sigma_2)/\bar{\sigma}$ along $x = b$, a vertical line through the crack tip.

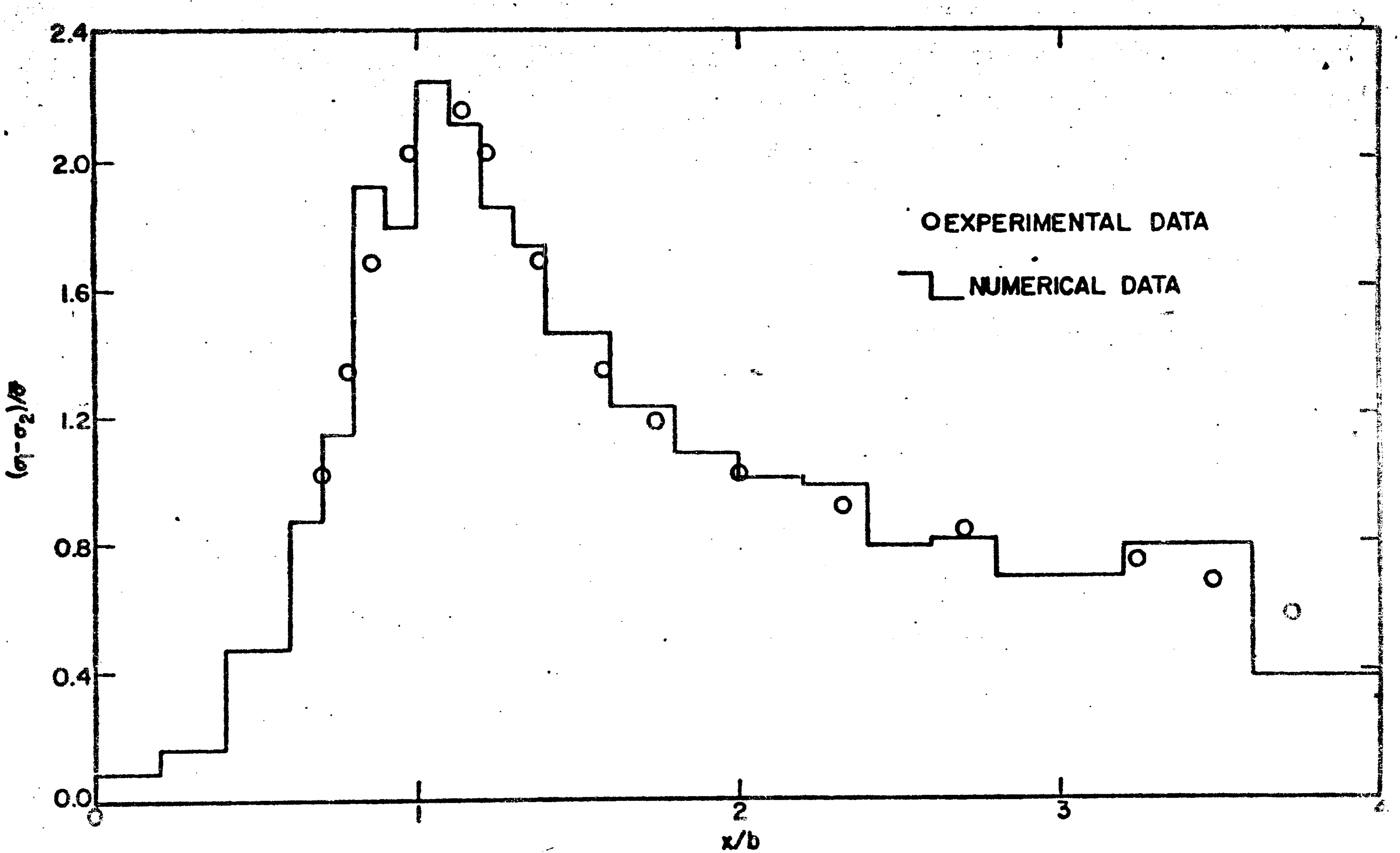

Figure 4.4 - Comparison of (elastic) numerical and photoelastic values of $2\tau_{max}/\sigma = (\sigma_1 - \sigma_2)/\sigma$ along $y = 0.4b$, a line above and parallel to the crack.

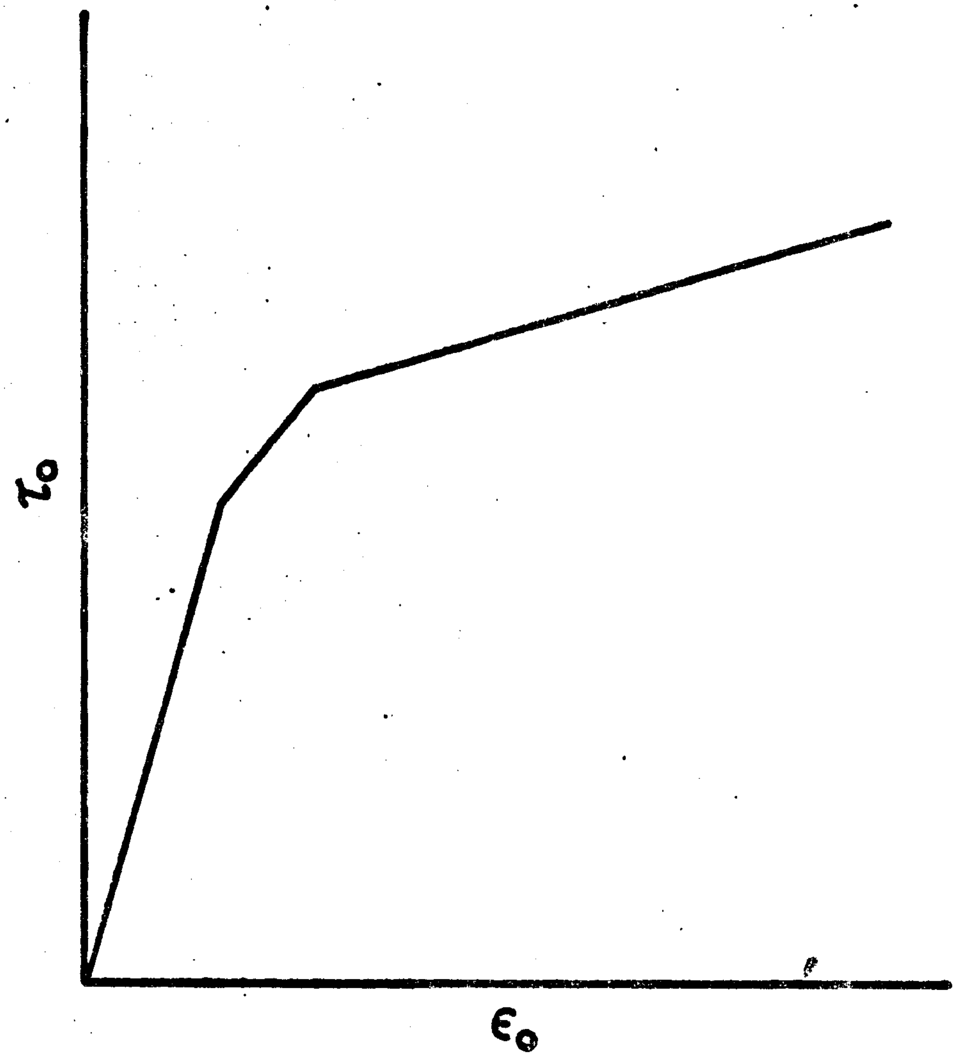

Figure 5.1 - Trilinear stress-strain curve.